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Constant curvature 2-spheres in $\mathbb{C}P^2$

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Abstract. In this note, for each $0 < R < 1$, we construct a continuous family of isometric immersions of the 2-sphere of constant curvature $1/R^2$, $S^2(R)$, into complex projective plane $\mathbb{C}P^2$ with the Fubini-Study metric.

Keywords: Euclidean sphere, isometric embedding, complex projective plane.

1. Introduction

We are interested in isometric embeddings of the 2-sphere of constant curvature $1/R^2$, $S^2(R)$, into complex projective plane $\mathbb{C}P^2$ with the Fubini-Study metric normalized so that the sectional curvature is between 1 and 4. It is easy to classify all such embeddings which are *complex* algebraic. In fact, if (z_0, z_1, z_2) denote homogeneous coordinates in $\mathbb{C}P^2$ and F is a homogeneous polynomial in those coordinates, then $F = 0$ has the topology of a sphere if and only if the degree of F is one or two. If the degree is one, then $F = 0$ describes a totally geodesic $\mathbb{C}P^1$ with constant curvature 4 ($R = 1/2$). If the degree is two, $F = 0$ is congruent to an sphere $az_0^2 + bz_1^2 + cz_2^2 = 0$ where a, b, c are real and $0 \le a \le b \le c$; that sphere will have constant curvature if and only if $a = b = c \neq 0$, in which case the curvature will be 2 ($R = \sqrt{2}/2$). It is interesting to observe also that $\mathbb{R}P^2$ represents an immersed sphere in $\mathbb{C}P^2$ of curvature 1 ($R = 1$).

In this note, for each $0 < R < 1$, we construct a continuous family of isometric immersions $S^2(R) \to \mathbb{C}P^2$. The procedure is a modification of an idea of Ferus and Pinkall [1] who constructed constant curvature

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2-spheres in the 4-sphere.

2. The Construction

Consider the fibration $S^1 \to S^5(1) \to \mathbb{C}P^2$ where we view

$$S^5(1) = \{(Z_0, Z_1, Z_2) \in \mathbb{C}^3 : |Z_0|^2 + |Z_1|^2 + |Z_2|^2 = 1\}$$

and S^1 acts diagonally. If $r_1,\ r_2 > 0,\ r_1^2 + r_2^2 = 1$, then $|Z_0|^2 = r_1^2$, $|Z_1|^2 + |Z_2|^2 = r_2^2$ defines an inclusion $S^1(r_1) \times S^3(r_2) \subset S^5(1)$ which is compatible with the fibration; therefore, it gives rise in the quotient to a homogeneous manifold M diffeomorphic to S^3. In fact, let $z_i = Z_i/Z_0$, $i : 1, 2$, be an affine chart on $U_0 = \{Z_0 \neq 0\}$. Then $M \subset U_0$ and the chart maps M onto $S^3(r) \subset \mathbb{C}^2$, where $r = r_1/r_2$. Moreover the metric induced on M can be expressed in this chart, after rescaling, as

$$\langle\langle a, b\rangle\rangle_z = \frac{4r^2}{(1+r^2)^2}[(1 + r^2|z_2|^2)\,\mathrm{Re}\{a_1\bar{b_1}\} + (1 + r^2|z_1|^2)\,\mathrm{Re}\{a_2\bar{b_2}\} - \\ - r^2\,\mathrm{Re}\{z_1\bar{z_2}(\bar{a_1}b_2 + \bar{b_1}a_2)\}]$$

where $z = (z_1, z_2) \in S^3$ and $a = (a_1, a_2)$, $b = (b_1, b_2) \in T_zS^3$. We propose to construct isometric immersions $S^2(R) \to (S^3, \langle\langle , \rangle\rangle)$.

The S^1 action $\rho : S^1 \times S^3 \to S^3$, $\rho_t(z_1, z_2) = (z_1, e^{it}z_2)$ is $\langle\langle , \rangle\rangle$-isometric and the orbit velocity square is

$$\begin{aligned}\left\|\frac{d}{dt}\rho_t(z_1, z_2)\right\|^2 &= \|(0, ie^{it}z_2)\|^2 \\ &= \frac{4r^2}{(1+r^2)^2}(1 + r^2|z_1|^2)|z_2|^2\end{aligned} \tag{1}$$

The orbit space is a 2-dimensional hemisphere, which can be parametrized by

$$X = \{(e^{i\phi}\cos\theta, \sin\theta) : 0 \le \phi \le 2\pi, 0 \le \theta \le \pi/2\}.$$

The boundary $\partial X = \{\theta = 0\}$ is the set of fixed points. Since

$$\frac{d}{dt}|_{t=0}\rho_t(z) = (0, iz_2),$$

the orbital metric $\{\cdot, \cdot\}$ on X is given by

$$\{a, a\} = \langle\langle a, a\rangle\rangle - \langle\langle a, (0, iz_2)\rangle\rangle^2 \|(0, iz_2)\|^{-2}$$

for $z \in X - \partial X$, $a \in T_z X$. Therefore,

$$E = \left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\} = \frac{4r^2}{1+r^2} \tag{2}$$

$$F = \left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right\} = 0 \tag{3}$$

$$G = \left\{ \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi} \right\} = \frac{4r^2 \cos^2 \theta}{(1+r^2)(1+r^2\cos^2\theta)} \tag{4}$$

If $2\pi l(\theta, \phi)$ is the length of the orbit through (θ, ϕ), then from (1)

$$l(\theta, \phi) = \frac{2r \sin \theta}{1+r^2} \sqrt{1 + r^2 \cos^2 \theta} \tag{5}$$

Assume $S^2(R) \to (S^3, \langle\langle , \rangle\rangle)$ is an isometric immersion which is S^1-equivariant with respect to the rotation of $S^2(R)$ around the z-axis and the ρ action on S^3. Consider a meridian $s \mapsto (R \sin s, R \cos s)$, $0 \le s \le \pi$. Its image in S^3 projects onto a curve $s \mapsto (\theta(s), \phi(s))$ in the orbit space X which starts and terminates at boundary points and satisfies

$$E\dot{\theta}^2 + G\dot{\phi}^2 = R^2 \tag{6}$$

$$l(\theta(s), \phi(s)) = R \sin s \tag{7}$$

Conversely, a curve in X satisfying (6) and (7) which meets ∂X perpendicularly at its endpoints generates an equivariant isometric immersion.

In order to construct such curves, differentiate the square of (7) to get

$$\dot{\phi}^2 = \frac{(1+r^2)^2}{4r^2} \frac{1+r^2\cos^2\theta}{(1+r^2-2r^2\sin^2\theta)^2} \frac{R^2 - \frac{4r^2}{(1+r^2)^2}\sin^2\theta(1+r^2\cos^2\theta)}{\cos^2\theta} \tag{8}$$

Consider θ as an independent variable, where possible, and use (6), (8), (2), (3) and (4) to get

$$\left(\frac{d\phi}{d\theta}\right)^2 = \frac{R^2}{G\dot{\theta}^2} - \frac{E}{G} = \tan^2\theta \frac{\Gamma}{4r^4\sin^4\theta - 4r^2(1+r^2)\sin^2\theta + R^2(1+r^2)^2} \tag{9}$$

where

$$\begin{aligned} \Gamma = {} & 4r^4(r^2 - R^2(1+r^2))\cos^4\theta + \\ & + 4r^2(2r^2 - R^2(1+r^2))\cos^2\theta + 4r^2 - R^2(1+r^2)^2. \end{aligned} \tag{9a}$$

Since $l(\theta,\phi) \leq 1$ (see (5)), we may assume $0 < R \leq 1$. Then (9) can be rewritten as

$$\left(\frac{d\phi}{d\theta}\right)^2 = g_1(\theta)g_2(\theta)g_3(\theta)$$

where

$$\begin{aligned} g_1(\theta) &= \frac{\sin^2\theta}{4r^2(1+r^2-r^2w^2-r^2\sin^2\theta)(w+\sin\theta)} \\ g_2(\theta) &= \frac{\Gamma}{\cos^4\theta}, \quad \text{with } \Gamma \text{ defined in } (9a) \\ g_3(\theta) &= \frac{\cos^2\theta}{w-\sin\theta} \end{aligned}$$

and

$$w = \sqrt{\frac{1+r^2}{2r^2}(1-\sqrt{1-R^2})}.$$

Note that $\theta_{R,r} = \arcsin w \in (0,\pi/2)$ is the smallest positive zero of the denominator of (9). Then it is easy to see that g_1 and g_3 are positive on $(0,\theta_{R,r})$, but we need to require $R < 1$ and

$$r \geq \frac{R}{2\sqrt{1-R^2}} \tag{10}$$

in order to have also g_2 positive on that interval. In fact, with those hypothesis, the g_i are positive increasing on $(0,\theta_{R,r})$ and $h = g_1g_2$ is bounded on $[0,\theta_{R,r}]$. Therefore

$$\phi(\theta) = \int_0^\theta \sqrt{h(t)}\frac{\cos t}{\sqrt{w-\sin t}}dt \tag{11}$$

defines a continuous function on $[0,\theta_{R,r}]$, smooth on $[0,\theta_{R,r})$. The corresponding curve in X can be reflected at the meridian $\phi = \phi(\theta_{R,r})$ (since the orbital metric is ϕ-independent) and gives a solution of (6), (7) starting and terminating at boundary points. Finally, $\phi = \phi(\theta)$ is an even function around $\theta = 0$ since its derivative is odd.

For each $R \in (0,1)$ this gives a family indexed by r (subject to (10)) of isometric immersions of $S^2(R)$ into $(S^3, \langle\langle,\rangle\rangle)$, and then into $\mathbb{CP}^2$ as well. One of those immersions will be an embedding if the generating curve is not self-intersecting, i.e. if $\phi(\theta_{R,r}) < \pi$.

Fix $R \in (0,1)$. Then we need $r \geq R/(2\sqrt{1-R^2})$. Since

$$\lim_{r\to\infty} \theta_{R,r} = \arcsin \lim_{r\to\infty} w = \arcsin \sqrt{\frac{1}{2}(1-\sqrt{1-R^2})}$$

and, for each $\theta \in (0, \arcsin\sqrt{\frac{1}{2}(1-\sqrt{1-R^2})})$ fixed,

$$\lim_{r\to\infty} \frac{d\phi}{d\theta}(\theta) = \infty,$$

we conclude that

$$\lim_{r\to\infty} \phi(\theta_{R,r}) = \infty$$

Similarly, as $R \to 1$ (which requires $r \to \infty$) the two smallest positive zeros of the denominator of (9) merge together into a double zero ($\theta = \pi/4$) and we get

$$\lim_{R\to 1} \phi(\theta_{R,r}) = \infty$$

Therefore, we do not have injectivity for R very close to 1. Moreover, in order to have injectivity for a given $R \in (0,1)$, we have to try with an r which is not too big. The monotonicity of h in (11) allows the estimate

$$\phi(\theta_{R,r}) \leq \int_0^{\theta'} \sqrt{h(t)}\frac{\cos t}{\sqrt{w-\sin t}}dt + 2\sqrt{h(\theta_{R,r})}\sqrt{w-\sin\theta'}$$

for $\theta' \in (0, \theta_{R,r})$. Numerical evaluation indicates then that we have injectivity at least for $R \in (0, .999999)$ if we choose r to be the smallest possible (cf. (10)).

References

[**1**] Ferus, D.; Pinkall, U., *Constant curvature* 2*-spheres in the* 4*-sphere*, Math. Z., **200**, (1989), 265-271.

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